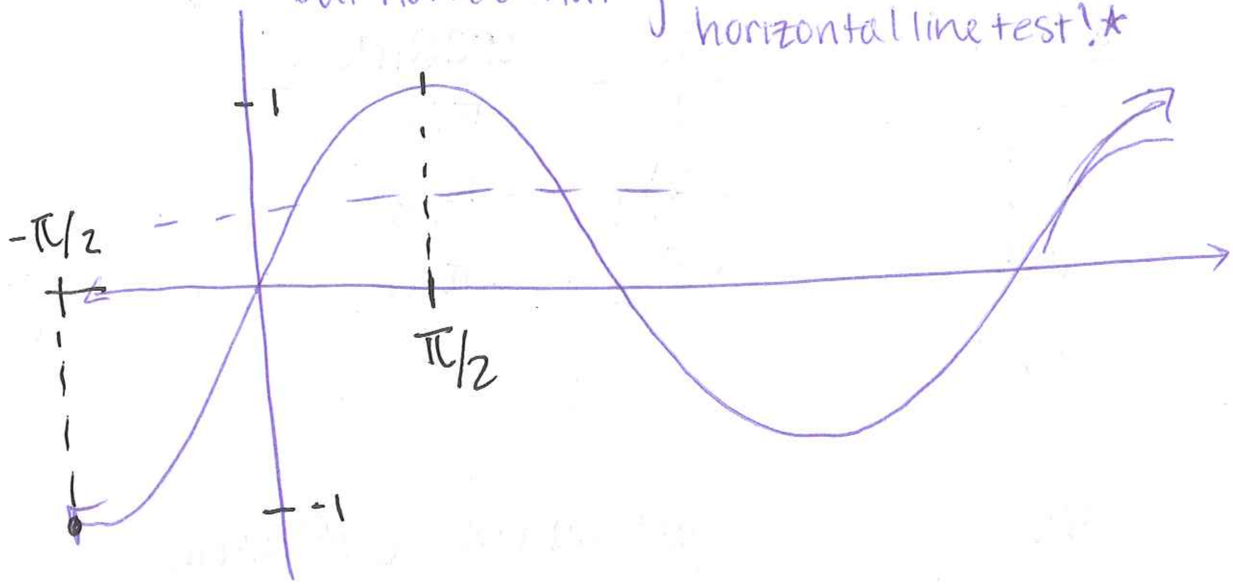
March 5, 2014

Inverse trig functions

$$\sin x = y \iff \arcsin y = x$$

$y = \arcsin x$ is inverse function of $y = \sin x$

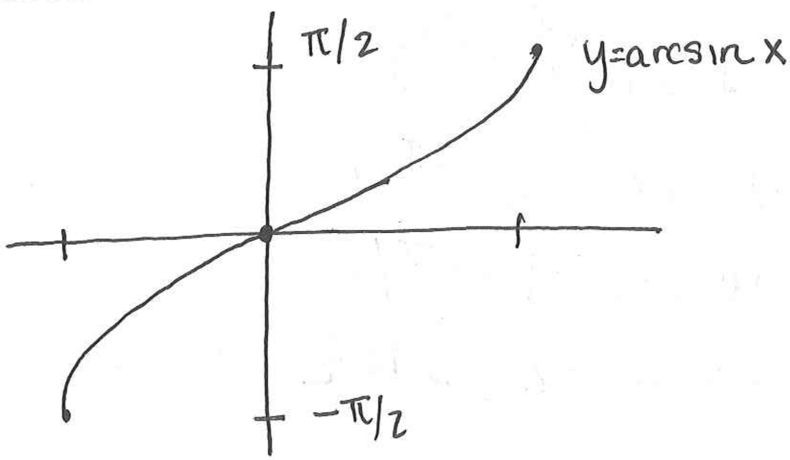
but notice that $y = \sin x$ doesn't pass the horizontal line test!



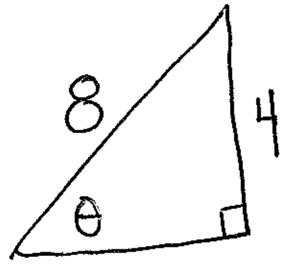
this means a true inverse wouldn't be a function.

we will find the inverse of $\sin x$ w/ restricted domain

For example: $y = \arcsin x$ has domain $[-1, 1]$



Idea:



What is the angle θ?

$$\underbrace{\sin\theta}_{30^\circ \text{ OR } \pi/6} = \frac{4}{8} = \frac{1}{2}$$

– So –

$$\arcsin \tfrac{1}{2} = \theta = \pi/6$$

Some values:

x	$\sin x$
$-\pi/2$	-1
$-\pi/3$	$-\sqrt{3}/2$
$-\pi/4$	$-\sqrt{2}/2$
$-\pi/6$	$-1/2$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

x	$\arcsin x$
-1	$-\pi/2$
$-\sqrt{3}/2$	$-\pi/3$
$-\sqrt{2}/2$	$-\pi/4$
$\vdots$	

just switch the columns.

Ex: $\arcsin \frac{\sqrt{3}}{2} = \pi/3$

$\arcsin -\frac{\sqrt{2}}{2} = -\pi/4$

$\arcsin -1 = -\pi/2$

$\arcsin -2 =$ D.N.E.

Ex: $\arcsin(\sin(\pi/3)) = \pi/3$

$\arcsin(\sin(3\pi/4))$

$= \arcsin(\sqrt{2}/2)$

$= \pi/4$ *Notice we didn't get $3\pi/4$ back - the restricted domain caused this*

(also, $\sin^{-1}x = \arcsin x$ notationwise)

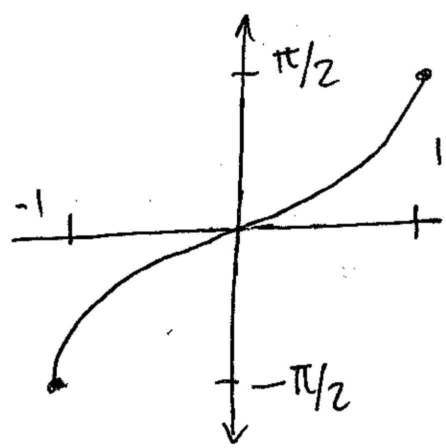
CAUTION: $\sin^{-1}x \neq \frac{1}{\sin x}$

Three Inverse Trig Functions

Arc Sine:

Domain: $[-1, 1]$

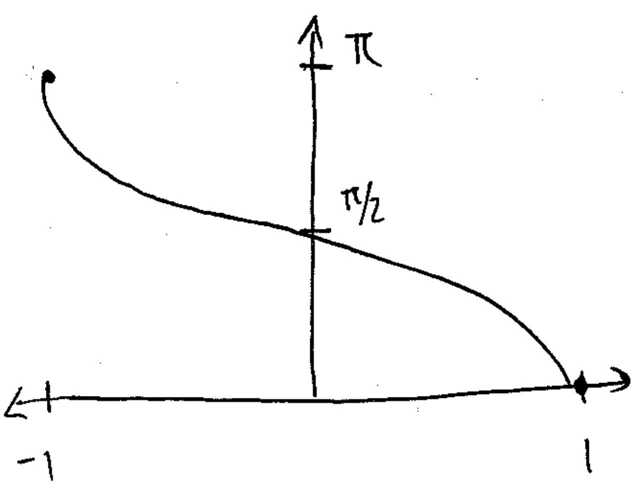
Range: $[-\pi/2, \pi/2]$



Arc Cosine:

Domain: $[-1, 1]$

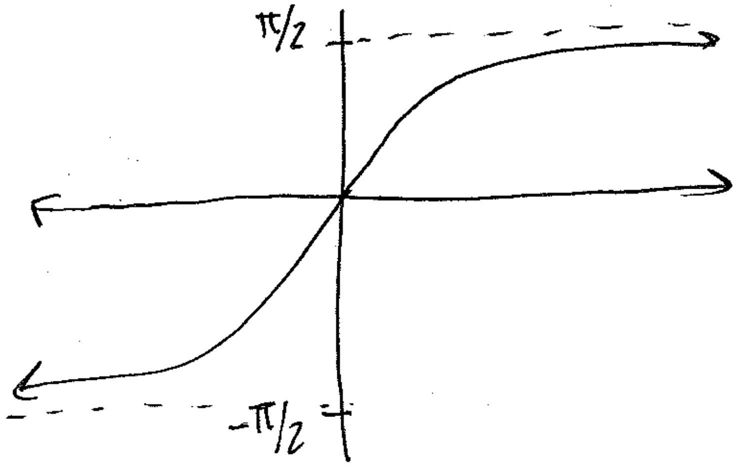
Range: $[0, \pi]$



ArcTangent

Domain: $(-\infty, \infty)$

Range: $(-\pi/2, \pi/2)$



Ex: $\arctan(0) = 0$

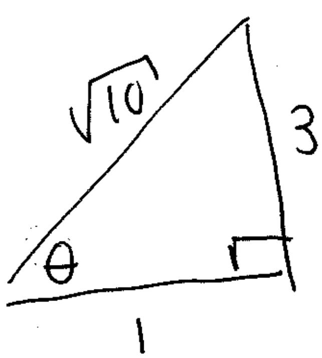
$\arctan(1) = \pi/4$ (when $\sin x = \cos x$, $\sqrt{2}/2 \Rightarrow \pi/4$)

$\arctan(-1) = -\pi/4$

$$\arctan(\underbrace{\tan(3\pi/4)}_{\frac{\sqrt{2}}{2} / -\frac{\sqrt{2}}{2}}) = \arctan(-1) = -\pi/4$$

Ex: Find $\cos(\arctan 3)$

trick: draw a right triangle:



want $\tan\theta = 3$

calculate hyp: $\sqrt{1+9} = \sqrt{10}$

so $\arctan 3 = \theta$

$\cos(\arctan 3) = \cos\theta = \frac{1}{\sqrt{10}}$

Derivatives of inverse trig functions

Derivative of arcsine:

$$y = \arcsin x$$

$$\sin y = x$$

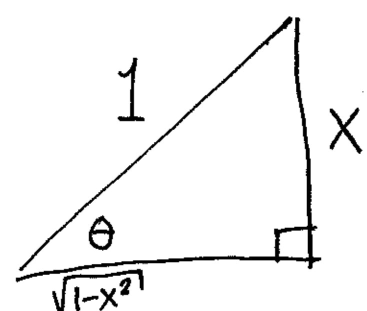
$$\frac{d}{dx}(\sin y) = \frac{d}{dx} x$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin x)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



$\sin\theta = x$

so $\arcsin x = \theta$

so $\cos(\arcsin x)$

$= \cos\theta = \sqrt{1-x^2}$

$$\boxed{\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}}$$

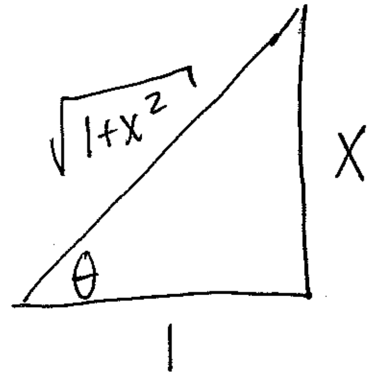
Similar method gets us that $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

Derivative of arctan x

$$y = \arctan x$$

$$\tan y = x$$

$$\frac{d}{dx} \tan y = \frac{d}{dx} x$$



$$\sec^2 y \frac{dy}{dx} = 1$$

$$\sec^2\theta = \left(\sqrt{1+x^2}\right)^2 = 1+x^2$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\sec^2(\arctan x)}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\boxed{\frac{d}{dx} \tan x = \frac{1}{1+x^2}}$$

General Forms: Let a be a constant

$\frac{d}{dx} \arcsin\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2-x^2}}$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C$
$\frac{d}{dx} \arccos\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2-x^2}}$	$\int \frac{-dx}{\sqrt{a^2-x^2}} = -\arcsin\left(\frac{x}{a}\right) + C = \arccos\left(\frac{x}{a}\right) + D$
$\frac{d}{dx} \arctan\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$

Examples:

(1) Find derivative of $y = e^{\arcsin x}$

$$\frac{dy}{dx} = e^{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{e^{\arcsin x}}{\sqrt{1-x^2}}$$

(2) Find derivative of $y = x^2 \cdot \arctan(e^x)$

$$\frac{dy}{dx} = 2x \cdot \arctan(e^x) + x^2 \cdot \frac{1}{(e^x)^2+1} \cdot e^x$$

$$= 2x \arctan(e^x + \frac{x^2 e^x}{1+e^{2x}}$$

(3) evaluate:

$$\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^2$$

$$= \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan 0$$

$$= \cancel{\frac{\sqrt{2}}{4}} \quad \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}$$

(4) evaluate:

u-sub: $u = e^x \quad du = e^x dx$

$$\int_{-1}^{0} \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int_{1/e}^{1} \frac{1}{\sqrt{1-u^2}}\,du = \arcsin u \Big|_{1/e}^{1}$$

$$= \arcsin 1 - \arcsin 1/e$$

$$\approx 1.19407\ldots$$

Interesting Integral
(doesn't have to do w/ derivative of inverse trig)

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

use u-sub
$u = \cos x$
$du = -\sin x \, dx$

$$= \int -\frac{1}{u} \, du$$

$$= -\ln u + C$$

$$= -\ln(\cos x) + C$$

$$= \ln(\cos x)^{-1} + C$$

$$\boxed{\int \tan x \, dx = \ln(\sec x) + C}$$

$$\int \frac{x}{3 + x^4} \, dx$$

$u = x^2 \quad du = 2x \, dx$

$$= \int \frac{1}{2} \cdot \frac{1}{3 + u^2} \, du = \frac{1}{2} \int \frac{1}{3 + u^2} \, du = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \arctan\left(\frac{u}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2\sqrt{3}} \arctan\left(\frac{x^2}{\sqrt{3}}\right) + C$$